RWE
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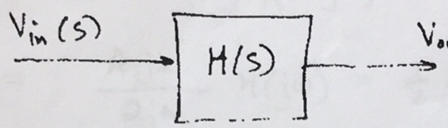
# Finding sinusoidal steady state using the Laplace transform

## Relationship between Laplace and phasors




$V_{out}(s) = H(s)\, V_{in}(s)$

Suppose $v_{in}(t) = A\cos(\omega t)\, u(t)$

Then $V_{in}(s) = A \dfrac{s}{s^2+\omega^2}$

$V_{out}(s) = A \dfrac{s}{s^2+\omega^2} H(s)$

Poles of $V_{in}(s)$ are at $s = \pm j\omega$ (on the imaginary axis);
$s^2+\omega^2 = (s+j\omega)(s-j\omega)$

Partial fraction expansion of $V_{out}(s)$:

$V_{out}(s) = \dfrac{K_1}{s-j\omega} + \dfrac{K_1^*}{s+j\omega}$ + terms arising from poles of $H(s)$

Let us assume that the poles of $H(s)$ all have negative real parts (all poles lie in left half of complex s plane). Then the poles $s_1, s_2, \ldots, s_k$ of $H(s)$ lead to inverse transform terms that are of the form $K e^{-s_n t}$. These represent transients that decay to zero after some time.

After all transient terms have decayed to zero, what remains is the <u>sinusoidal steady state response</u>, given by the first two terms in the above partial fraction expansion of $V_{out}(s)$. Let's work out $K_1$ and $v_{out}(t)$.

To find $K_1$, multiply by $s-j\omega$ and let $s \to j\omega$:

$$K_1 = (s-j\omega) \frac{A\,s}{(s+j\omega)(s-j\omega)} H(s) \Big|_{s \to j\omega}$$

$$= \frac{A\,j\omega}{2j\omega} H(j\omega) = \tfrac{1}{2} A\, H(j\omega)$$

$$K_1^* = \tfrac{1}{2} A\, H^*(j\omega)$$

The sinusoidal steady-state response is therefore

$$V_{OUT}(s) = \frac{\frac{1}{2} A\, H(j\omega)}{s-j\omega} + \frac{\frac{1}{2} A\, H^*(j\omega)}{s+j\omega}$$

Let's write $H(j\omega)$ in polar form; define

$$H(j\omega) = \|H(j\omega)\|\, e^{j\theta} \qquad \text{where } \theta = \angle H(j\omega)$$

note $H^*(j\omega) = \|H(j\omega)\|\, e^{-j\theta}$

So

$$V_{out}(s) = \tfrac{1}{2} A \|H(j\omega)\| \cdot \left[ \frac{e^{j\theta}}{s-j\omega} + \frac{e^{-j\theta}}{s+j\omega} \right]$$

inverse transform is

$$v_{out}(t) = \tfrac{1}{2} A \|H(j\omega)\| \cdot \left[ e^{j\theta} e^{j\omega t} + e^{-j\theta} e^{-j\omega t} \right]$$

$$= A \|H(j\omega)\| \cdot \frac{e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}}{2}$$

$$v_{out}(t) = A \|H(j\omega)\| \cos(\omega t + \theta)$$

Conclusion: If we ① apply a sinusoid to the input of a circuit having transfer function $H(s)$, and ② wait for all transients to decay to zero, then

(i) the output is a sinusoid having the same frequency $\omega$ as the input

(ii) (Amplitude of output sinusoid) = $\|H(j\omega)\| \cdot$ (Amplitude of input sinusoid)

(iii) (Phase of output sinusoid) = $\angle H(j\omega)$ + (Phase of input sinusoid)

Analysis of the circuit to find the magnitude and phase of the transfer function is equivalent to use of the phasor method.

This explains why we can let $s \rightarrow j\omega$ to find sinusoidal steady state.