$$f(s) = \int_0^\infty e^{-st} F(t) dt \qquad s = \sigma + i\omega$$

| Time Domain F(t) | Frequency Domain f(s) |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t$ | $\frac{1}{s^2}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\frac{sin(at)}{a}$ | $\frac{1}{s^2+a^2}$ |
| $\frac{sinh(at)}{a}$ | $\frac{1}{s^2-a^2}$ |

**Properties of the LaPlace Transform:** $$f(s) = \int_0^\infty e^{-st} F(t) dt$$

| Time Domain (t) | Frequency Domain (s) |
|---|---|
| $aF(t) + bG(t)$ | $af(s) + bf(s)$ |
| $F(t) * G(t)$ | $f(s)g(s)$ |
| $F'(t) = \frac{dF}{dt}$ | $sf(s) - F(t=0)$ |
| $F''(t) = \frac{d^2F}{dt^2}$ | $s^2 f(s) - sF(t=0) - F'(t=0)$ |
| $e^{at}F(t)$ | $f(s-a)$ |
| $F(t) * G(t)$ | $f(s)g(s)$ |

*Convolution Integral:* $$F(t) * G(t) = \int_{-\infty}^{\infty} F(\tau) G(t-\tau) d\tau$$

**Laplace Transform Example:** **Simple Harmonic Oscillator**

$$mX''(t) = -kX(t) \qquad 2X'' = -8X$$

*Initial Conditions:*

$$X(0) = 10; X'(0) = 0$$

*Simplify*

$$X'' + 4X = 0$$

*Take the Laplace Transform*

$$s^2x(s) - 10s + 4x(s) = 0$$

*Solve for x(s)*

$$x(s) = \frac{10s}{s^2 + 4}$$

*Take inverse Laplace Transform to get X(t)*

$$X(t) = 10cos(2t)$$

*oscillatory behavior*

**Laplace Transform Example:** **Damped Harmonic Oscillator**

$$mX''(t) = -kX(t) - cX'(t) \qquad 2X'' = -8X - 8X'$$

*Initial Conditions:*

$$X(0) = 10; X'(0) = 0$$

*Simplify*

$$X'' + 4X' + 4X = 0$$

*Take the Laplace Transform*

$$s^2x(s) - 10s + 4(sx(s) - 10) + 4x(s) = 0$$

*Solve for x(s)*

$$x(s) = \frac{10(s+2)+20}{(s+2)^2} = \frac{10}{(s+2)} + \frac{20}{(s+2)^2}$$

*Take inverse Laplace Transform to get X(t)*

$$X(t) = 10e^{-2t} + 20te^{-2t}$$

*overdamped*

**Laplace Transform Example:** **Forced Harmonic Oscillator**

$$mX''(t) = -kX(t) + F(t)$$

$$2X'' = -8X + F(t)$$ *Initial Conditions:*

$$F(t) = F_0 cos(\omega t)$$

$$X(0) = 10; X'(0) = 0$$

*Simplify* $$X'' + 4X = \frac{F_0}{2} cos(\omega t)$$

*Take the Laplace Transform* $$s^2 x(s) - 10s + 4x(s) = \frac{F_0}{2} \frac{s}{s^2 + \omega^2}$$

*Solve for x(s)* $$x(s) = \frac{10s}{s^2 + 4} + \frac{F_0}{2} \frac{s}{(s^2 + 4)(s^2 + \omega^2)}$$

*Take inverse Laplace Transform to get X(t)*

$$X(t) = 10cos(2t) + \frac{F_0}{2(\omega^2 - 4)}(cos(2t) - cos(\omega t))$$

$\omega^2 \neq 4$ *driven off resonance*

$$X(t) = 10cos(2t) + \frac{F_0}{8}(tsin(2t))$$

$\omega^2 = 4$ *driven on resonance*